we may take either $y_r = -a$, whence $x = (b + 1)/10$; or $y_r = b$, $x = (-a + 1)/10$. Thus solutions go in pairs, the x of either solution being obtained by adding 1 to the y_r of the other and dividing by 10. There are, then, always an even number of solutions, and always at least two, viz., those where $y_r = -1$, $x = 10^{r-1}(n + 1)$; $y_r = 10^r(n + 1) - 1$, $x = 0$.

Finally it may be noted that the more general equation

$$ax^r - bxy + y - c = 0$$

may be solved by similar methods. Here we should have to pick out such divisors of $b^rc - a$ as are $\equiv \pm 1$ (mod. b). It may be interesting to apply the method to an example. Let us solve

$$x^4 - 10xy - 22 + y = 0.$$

We have to solve

$$(10x - 1)y_4 = -219999 = -3 \times 13 \times 5641$$

(5641 prime). The values of y_4 with the corresponding solutions of our equation are:

$$y_4 = -5641, \quad x = 4, \quad y = 6; \quad y_4 = -1, \quad x = 22000 \text{ (y a very large number)};$$

$$y_4 = 39, \quad x = -564, \quad y = -17937434; \quad y_4 = 219999, \quad x = 0, \quad y = 22.$$

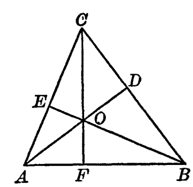
247 (Number Theory) [June, 1916]. Proposed by NORMAN ANNING, Chilliwack, B. C.

To dissect the triangle whose sides are 52, 56, 60 into three Heronian triangles by lines drawn from the vertices to a point within.

The word Heronian is used in the sense of the German Heronische (Wertheim, *Anfangsgründe d. Zahlenlehre*, p. 140) to describe a triangle whose sides and area are integral.

SOLUTION BY FRANK IRWIN, University of California.

The orthocenter, O, may be taken as the required point. Let ABC be the triangle with $a = 60$, $b = 52$, $c = 56$; and let the feet of the perpendiculars from A, B, C on the opposite sides be D, E, F, respectively. Then the various lines in the figure, calculated as indicated below, are: $BD = 168/5$, $DC = 132/5$, $CE = 396/13$, $EA = 280/13$, $AF = 20$, $FB = 36$; $AO = 25$, $BO = 39$, $CO = 33$. Finally, area $BOC = 594$, area $COA = 330$, and area $AOB = 420$; so that the sides and areas of these three triangles are integral, as asserted.



The explanation of these facts depends on the following proposition: If the sides and area of the triangle ABC are rational, the same is true of the triangles BOC, COA, AOB. (Then by multiplying the dimensions of the figure by a suitable integer everything can be made integral.) For the three altitudes are rational, as also the radius r of the inscribed circle (since rs = area). Thus $\tan A/2$ is rational, and so, then, are $\cos^2 A/2$ and $\cos A$. Therefore, $AF = b \cos A$ is rational, and similarly, FB, BD, etc. Then the triangle AOF is rational (that is, has rational sides), since one of its sides, AF, is rational, and it is similar to the rational triangle ABD.

2678 [February, 1918]. Problem proposed by C. F. GUMMER, Queen's University, Canada.

Find necessary and sufficient conditions that the roots of the equation $x^{n+1} + a_1x^n + a_2x^{n-1} + \cdots + a_{n+1} = 0$ may be all real and separated by the roots of $x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n = 0$.

SOLUTION BY THE PROPOSER.

Consider the equations

$$f(x) \equiv x^{n+1} + a_1x^n + \cdots = 0, \tag{1}$$

$$g(x) \equiv x^n + b_1x^{n-1} + \cdots = 0, \tag{2}$$

$$R_1(x) \equiv c_0x^{n-p} + c_1x^{n-p-1} + \cdots = 0, \tag{3}$$

$$R_2(x) \equiv d_0x^{n-p-q} + d_1x^{n-p-q-1} + \cdots = 0, \tag{4}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

where $R_1(x)$ is the remainder with sign changed on dividing $f(x)$ by $g(x)$, $R_2(x)$ has the same relation to $g(x)$ and $R_1(x)$, etc.

That the roots of (1) may be real and separated by those of (2), it is necessary and sufficient that

(a) the roots of $g(x)$ shall be real and distinct (such as $\beta_1 < \beta_2 < \cdots < \beta_n$) and

(b) $f(-\infty), f(\beta_1), \cdots, f(\beta_n), f(\infty)$ shall have alternate signs.

Now $f(\beta_i) = -R_1(\beta_i)$, and $R_1(x)$ cannot change sign more than $n-1$ times. Hence, (a) and (b) are equivalent to the conditions that c_0 shall be positive, and that the roots of (2) are all real and separated by those of (3) (implying that $p = 1$). These are in turn equivalent to the conditions that c_0, d_0 shall be positive, $p = q = 1$, and the roots of (3) real and separated by those of (4), and so on.

Hence, a set of necessary and sufficient conditions is

$$p = q = \cdots = 1;\ c_0,\ d_0,\ \cdots, \text{ all positive.}$$

It may be shown that these are equivalent to the conditions

$$\begin{vmatrix} 1 & b_1 & b_2 \\ 1 & a_1 & a_2 \\ 0 & 1 & b_1 \end{vmatrix}, \qquad \begin{vmatrix} 1 & b_1 & b_2 & b_3 & b_4 \\ 1 & a_1 & a_2 & a_3 & a_4 \\ 0 & 1 & b_1 & b_2 & b_3 \\ 0 & 1 & a_1 & a_2 & a_3 \\ 0 & 0 & 1 & b_1 & b_2 \end{vmatrix}, \qquad \cdots$$

all positive. For the calculation of the remainders see papers by E. B. Van Vleck (*Annals of Mathematics*, 1899, pp. 1–13) and A. J. Pell and R. L. Gordon (*Annals of Mathematics*, June, 1917, pp. 188–193).

2701 [May, 1918]. Proposed by E. H. WORTHINGTON, University of Pennsylvania.

Find the sum of the infinite series

$$\frac{1}{5} + \frac{1\cdot 2}{5\cdot 7} r + \frac{1\cdot 2\cdot 3}{5\cdot 7\cdot 9} r^2 + \cdots + \frac{n!}{5\cdot 7\cdot 9\cdots(2n+3)} r^{n-1} + \cdots$$

Verify your result for $r = 0$ and $r = 1$.

SOLUTION BY A. M. HARDING, University of Arkansas.

It can be easily shown that the following equation holds for all integral values of $n \geqq 0$:

$$\int_0^1 t^{1/2}(1-t)^{n+1}dt = \frac{2(n+1)}{2n+5}\int_0^1 t^{1/2}(1-t)^n dt.$$

Multiplying both sides by x^n and substituting $n = 0. 1, 2, \cdots (n-1)$, gives

$$\frac{3}{4}\int_0^1 t^{1/2}(1-t)dt = \tfrac{1}{5},$$

$$\frac{3}{4}\int_0^1 t^{1/2}(1-t)^2 x dt = \frac{1\cdot 2}{5\cdot 7} 2x,$$

$$\frac{3}{4}\int_0^1 t^{1/2}(1-t)^3 x^2 dt = \frac{1\cdot 2\cdot 3}{5\cdot 7\cdot 9}(2x)^2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{3}{4}\int_0^1 t^{1/2}(1-t)^n x^{n-1} dt = \frac{n!}{5\cdot 7\cdot 9\cdots(2n+3)}(2x)^{n-1}.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Letting $x = r/2$ and adding, we have

$$\frac{3}{4}\int_0^1 t^{1/2}(1-t)[1 + (1-t)x + \cdots + (1-t)^{n-1}x^{n-1} + \cdots]dt$$
$$= \frac{1}{5} + \frac{1\cdot 2}{5\cdot 7} r + \cdots + \frac{n!}{5\cdot 7\cdot 9\cdots(2n+3)} r^{n-1} + \cdots.$$

The series in the right member of this equation converges if $r < 2$. In this case, the series under the integral converges uniformly to the sum $\dfrac{1}{1-(1-t)r/2}$. Hence, the sum of the given series is given by